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A generalisation of the Luttinger model

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Abstract. A generalised Luttinger model is proposed and studied in the spinless and spin-dependent cases. The solution of the back-scattering and Umklapp scattering problems are obtained using a new strategy. For the standard Luttinger model the results already known are obtained again.

1. Introduction

1.1. The general framework

The bifermionic model of the one-dimensional electron gas, proposed by Luttinger (1963) and exactly solved by Mattis and Lieb (1965), is an exactly soluble model, whose properties are extensively studied and well understood (see, e.g., Sòlyom 1979, Emery 1979, Bohr 1981). In the simplest variant, that of the spinless fermions, the model is described by the Hamiltonian

$$H_{\mathrm{LM}} = H_0 + H_{\mathrm{i}} \tag{1.1}$$

where H_0 is the kinetic part given by

$$H_0 = \int_0^L \mathrm{d}x[\psi_1^+(x)\hat{p}\psi_1(x) - \psi_2^+(x)\hat{p}\psi_2(x)] \tag{1.2}$$

and H_{i} is the interaction part, which describes a forward-scattering process:

$$H_{\mathrm{i}} = \int_0^L \mathrm{d}x\,\mathrm{d}y\ \psi_1^+(x)\psi_1(x)V(x-y)\psi_2^+(y)\psi_2(y). \tag{1.3}$$

The fermionic field operator may be written as

$$\psi_j(x) = \frac{1}{\sqrt{L}}\sum_k \exp(\mathrm{i}kx)\,a_{jk}. \tag{1.4}$$

The Fourier components of the particle number density operator

$$N_j(x) = \psi_j^+(x)\psi_j(x) = \frac{1}{L}\sum_k \rho_j(k)\exp(-\mathrm{i}kx) \tag{1.5}$$

are

$$\rho_j(k) = \sum_p a_{j,p+k}^+ a_{j,p}. \tag{1.6}$$

These momentum space density operators $\rho_j(k)$ satisfy bosonic commutation relations:

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$$[\rho_1(-k), \rho_1^+(-k)] = [\rho_2(k), \rho_2^+(k)] = kL/2\pi \qquad k>0 \tag{1.7}$$

and the whole Hamiltonian may be expressed in bosonic terms:

$$H_{\mathrm{LM}} = \frac{2\pi v_{\mathrm{F}}}{L} \sum_{k>0} [\rho_1^+(-k)\rho_1(-k) + \rho_2^+(k)\rho_2(k)] + \frac{2\pi\lambda}{L} \sum_{k>0} [\rho_1(-k)\rho_2(k) + \rho_2^+(k)\rho_1^+(-k)]. \tag{1.8}$$

The Hamiltonian (1.8) may be diagonalised by a Mattis–Lieb (1965) transformation (MLT), which is a Bogoliubov transformation of the bosonic operators

$$\begin{aligned} \tilde{\rho}_1(-k) &= \rho_1(-k)\cosh\varphi_k + \rho_2^+(k)\sinh\varphi_k \\ \tilde{\rho}_2(k) &= \rho_1^+(-k)\sinh\varphi_k + \rho_2(k)\cosh\varphi_k. \end{aligned} \tag{1.9}$$

1.2. *The limitations of the MLT*

In the form (1.3), the model leaves out an important interaction, the back-scattering (BS) process, which, in this simple case of the spinless fermions, may be written as

$$H_{\mathrm{BS}} = \int_0^L \mathrm{d}x[W(x)\psi_1(x)\psi_2^+(x) + \mathrm{HC}] \tag{1.10}$$

(HC means the Hermitian conjugated part) and may be interpreted here as an impurity scattering. The MLT modifies in a complicated manner the $\psi_1\psi_2^+$ term and, as a consequence, the model characterised by the Hamiltonian

$$H = H_{\mathrm{LM}} + H_{\mathrm{BS}} \tag{1.11}$$

is not exactly soluble. The same difficulty persists, evidently, in the spin-dependent problem, where the MLT modifies the BS and Umklapp scattering (US) terms in a complicated manner.

1.3. *Problem*

Does a 'simple' modification of the Hamiltonian (1.11) exist which admits an exact solution?

1.4. *Answer*

The answer is the following: *if* the bosonised part of the model has an interaction term of the form

$$H_{\mathrm{i}}' = \frac{2\pi}{L} \sum_{k>0} [(\lambda + \mathrm{i}\mu)\rho_1^+(-k)\rho_2^+(k) + \mathrm{HC}] \tag{1.12}$$

and *if* the coupling satisfies the relation

$$\mu^2 = 2\lambda(v_{\mathrm{F}} - \lambda) \tag{1.13}$$

then the transformation

$$\begin{aligned}\tilde{\rho}_1(-k) &= (1-\mathrm{i}\gamma)\rho_1(-k) - \mathrm{i}\gamma\rho_2^\dagger(k)\\ \tilde{\rho}_2(k) &= -\mathrm{i}\gamma\rho_1^\dagger(-k) + (1-\mathrm{i}\gamma)\rho_2(k)\end{aligned} \tag{1.14}$$

diagonalises the generalised Luttinger model (GLM) defined by the Hamiltonian

$$\begin{aligned}H_{\mathrm{GLM}} = \frac{2\pi v_{\mathrm{F}}}{L}\sum_{k>0}[\rho_1^\dagger(-k)\rho_1(-k) + \rho_2^\dagger(k)\rho_2(k)]\\ + \frac{2\pi}{L}\sum_{k>0}[(\lambda+\mathrm{i}\mu)\rho_1^\dagger(-k)\rho_2^\dagger(k) + \mathrm{HC}]\end{aligned} \tag{1.15}$$

and leaves unchanged the BS term (1.10), *if* γ is given by the relation

$$\gamma^2 = \lambda/2(v_{\mathrm{F}} - \lambda). \tag{1.16}$$

The Hamiltonian which admits an exact solution is $H_{\mathrm{GLM}} + H_{\mathrm{BS}}$. These questions were introduced and discussed previously (Bârsan 1985).

1.5. The spin-dependent case

We shall define a spin-dependent Luttinger model, for which the small-momentum-transfer terms are a spin-dependent variant of (1.15) and the large-momentum-transfer terms are the standard ones. We shall prove that the transformation (1.14), used in conjunction with the MLT, allows us to obtain an exact solution of the model, if some restrictions are imposed on the couplings. This solution is similar to those obtained by Luther and Emery (1974) and by Emery *et al* (1975).

2. A canonical transformation which leaves the back-scattering term unchanged

As was demonstrated (Bârsan 1985), the generalised Luttinger Hamiltonian (1.15) may be diagonalised by a transformation $\mathcal{T}$, defined by (1.14), which also has the property that it leaves unchanged the BS term (1.10), if the following conditions are fulfilled:

$$\begin{aligned}2\gamma^2(v_{\mathrm{F}} - \lambda) - 2\mu\gamma + \lambda = 0\\ 2\gamma(v_{\mathrm{F}} - \lambda) - \mu = 0.\end{aligned} \tag{2.1}$$

Equivalently,

$$\mu = 2\gamma(v_{\mathrm{F}} - \lambda) \tag{2.2}$$

and

$$\gamma^2 = \lambda/2(v_{\mathrm{F}} - \lambda). \tag{2.3}$$

Therefore,

$$0 < \lambda < v_{\mathrm{F}}. \tag{2.4}$$

It is easy to see that the above restrictions allow the existence of two solutions:

$$\gamma = \pm\sqrt{\lambda/2(v_{\mathrm{F}} - \lambda)} \qquad \operatorname{sgn}\gamma = \operatorname{sgn}\mu$$

which reflects the freedom of changing $\gamma \to -\gamma$ in (1.14), with the modulus fixed by (2.3). In both cases,

$$\mu^2 = 2\lambda(v_{\mathrm{F}} - \lambda). \tag{2.5}$$

If we prefer to write the complex couplings as

$$\lambda + \mathrm{i}\mu = \Lambda\exp(\mathrm{i}\theta) \tag{2.6}$$

the conditions (2.2) and (2.3) become

$$\Lambda = (2v_{\mathrm{F}}\cos\theta)/(1+\cos^2\theta) \qquad \cos\theta > 0. \tag{2.7}$$

This formula also includes the condition (2.4), because

$$\theta < \lambda \leqslant \Lambda \leqslant \Lambda_{\max} = v_{\mathrm{F}}.$$

Geometrically, the constraint (2.5), written in the form

$$(\lambda - v_{\mathrm{F}}/2)^2/(v_{\mathrm{F}}/2)^2 + \mu^2/(v_{\mathrm{F}}/\sqrt{2})^2 = 1 \tag{2.8}$$

represents an ellipse in the (λ, μ) plane, having its centre at $(v_{\mathrm{F}}/2, 0)$ and the semiaxes $v_{\mathrm{F}}/2$, $v_{\mathrm{F}}/\sqrt{2}$ (figure 1).

Such a transformation can diagonalise a Hamiltonian similar to (1.11):

$$\mathscr{H} = H_{\mathrm{GLM}}(\lambda, \mu) + H_{\mathrm{BS}} \tag{2.9}$$

if the couplings satisfy the restriction (2.5). Indeed, the $\mathscr{T}$ transformation leaves unchanged H_{BS} and diagonalises H_{GLM}; this last term can be written as a bilinear form in the fermionic operators. So, (2.9) becomes a quadratic form, the diagonalisation of which is well known.

3. A more general canonical transformation

It is clear that it is impossible to pass from the $\mathscr{T}$ transformation to the MLT or vice versa, choosing a particular value of the parameters involved in their expressions.

3.1. Problem

Does a more general transformation, containing $\mathscr{T}$ and MLT as particular cases, exist?

Let us remember that the MLT is generated by the Hermitian operator

$$S_{\mathrm{ML}} = \frac{2\pi i}{L} \sum_{p>0} \frac{\varphi(p)}{p} [\rho_1^+(-p)\rho_2^+(p) - \rho_1(-p)\rho_2(p)] \tag{3.1}$$

with $\varphi(p)$ a real and even function, and that the $\mathscr{T}$ transformation is generated by (Bârsan 1985)

$$S_{\mathscr{T}} = \frac{2\pi}{L} \sum_{p>0} \frac{\gamma_p}{p} [\rho_1^+(-p)\rho_1(-p) + \rho_2^+(p)\rho_2(p) + \rho_1(-p)\rho_2(p) + \rho_1^+(-p)\rho_2^+(p)] \tag{3.2}$$

(the momentum dependence of $\varphi(p)$, γ_p is relevant for convergence only; so it is often relaxed).

3.2. Answer

A 'sufficiently general' canonical transformation is that generated by the operator

$$S(\alpha, \beta) = \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \{\alpha_p[\rho_1^+(-p)\rho_1(-p) + \rho_2^+(p)\rho_2(p)]$$
$$+ \beta_p \rho_1(-p)\rho_2(p) + \beta_p^* \rho_1^+(-p)\rho_2^+(\rho)\}. \tag{3.3}$$

Indeed,

$$S(\gamma, \gamma) = S_{\mathscr{T}} \tag{3.4}$$

$$S(0, -\mathrm{i}\varphi) = S_{\mathrm{ML}}. \tag{3.5}$$

3.3. The action of the $S(\alpha, \beta)$ transformation upon the density operators

The density operators

$$\tilde{\rho} = \exp[\mathrm{i}S(\alpha, \beta)]\rho \exp[-\mathrm{i}S(\alpha, \beta)]$$

transformed by $S(\alpha, \beta)$ are given by

$$\begin{aligned} \tilde{\rho}_1(-p) &= a_p \rho_1(-p) + b_p \rho_2^+(p) \\ \tilde{\rho}_2(p) &= b_p \rho_1^+(-p) + a_p \rho_2(p) \end{aligned} \tag{3.6}$$

where

$$a_p = \cos d_p - \mathrm{i}\alpha_p[(\sin d_p)/d_p] \tag{3.7}$$

$$b_p = -\mathrm{i}[(\sin d_p)/d_p]\beta_p^* \tag{3.8}$$

$$d_p^2 = \alpha_p^2 - \beta_p^* \beta_p. \tag{3.9}$$

As a consequence of the canonicity of the transformation (3.3),

$$|a_p|^2 - |b_p|^2 = 1. \tag{3.10}$$

3.4. The action of $S(\alpha, \beta)$ upon the GLM

It is easy to verify that

$$\begin{aligned} &\exp[\mathrm{i}S(\alpha, \beta)]\, H_{\mathrm{GLM}}(v_{\mathrm{F}}, \lambda, \mu) \exp[-\mathrm{i}S(\alpha, \beta)] \\ &\quad = \frac{2\pi}{L} \sum_{p>0} [v_{\mathrm{F}}(|a_p|^2 + |b_p|^2) + \lambda_p(a_p^* b_p^* + a_p b_p) + \mathrm{i}\mu_p(a_p^* b_p^* - a_p b_p)] \\ &\quad \times [\rho_1^+(-p)\rho_1(-p) + \rho_2^+(p)\rho_2(p)] + \frac{2\pi}{L}\sum_{p>0} \{[2v_{\mathrm{F}} a_p^* b_p + (\lambda_p + \mathrm{i}\mu_p)a_p^{*2} \\ &\quad + (\lambda_p - \mathrm{i}\mu_p)b_p^2]\rho_1^+(-p)\rho_2^+(p) + \mathrm{HC}\} + W \end{aligned} \tag{3.11}$$

where W is the vacuum renormalisation energy, the value of which is unimportant in this context.

3.5. The $S(\alpha, \beta)$ transformation can diagonalise the GLM

Let us put

$$a = |a| \exp(\mathrm{i}\theta_a) \qquad b = |b| \exp(\mathrm{i}\theta_b). \tag{3.12}$$

The off-diagonal term from (3.11) will disappear if

$$|a| = 1/\sqrt{1 - A^2} \qquad |b| = A/\sqrt{1 - A^2} \tag{3.13}$$

$$\theta_a + \theta_b = \theta - \pi \tag{3.14}$$

where A is defined by

$$A = (1 - \sqrt{1 - x^2})/x \qquad x = \Lambda/v_{\mathrm{F}}. \tag{3.15}$$

The restrictions (3.13) and (3.14) may be also written as

$$|a| = 1/\sqrt{1 - A^2} \qquad b = -a^* A \exp(\mathrm{i}\theta). \tag{3.16}$$

So the diagonalisation conditions determine the coefficients a and b up to a phase

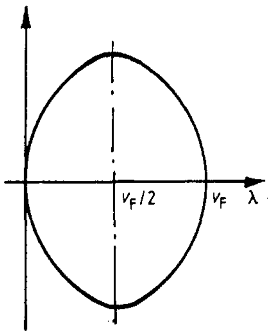

Figure 1. The ellipse characterised by equation (2.8), describing the connection between the couplings λ, μ, for which the $\mathcal{T}$ transformation (1.14) diagonalises the generalised Luttinger Hamiltonian.

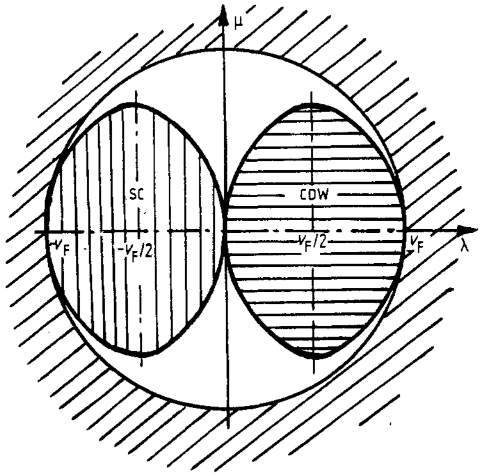

Figure 2. The phase diagram of the generalised Luttinger Hamiltonian.

θ_a or θ_b; this uncertainty is unimportant, because the computation of the physically interesting quantities does not require separate knowledge of θ_a and θ_b. Similarly, it is not important that we have not expressed α and β—only a and b—as functions of potential. These formulae will be useful when we compute the correlation functions of the GLM with the bosonisation method.

For large couplings, the diagonalisation conditions become meaningless (see (3.15)). This reflects the fact that the large-coupling regime cannot be connected to the free ground state by a unitary transformation.

If $S(\alpha, \beta)$ diagonalises the GLM, i.e. if the values of the parameters α and β are given by equations (3.7)–(3.10) and (3.13)–(3.16), the renormalised Fermi velocity is

$$\tilde{v}_F = v_F\sqrt{1-x^2}. \tag{3.17}$$

At this stage, λ and μ (Λ and θ) are free of any constraint.

3.6. Particular cases

3.6.1. *MLT, attractive case.* In this case

$$\lambda < 0 \qquad \mu = 0 \qquad \theta = \pi. \tag{3.18}$$

If we choose $\theta_a = 0$, (3.14) implies that $\theta_b = 0$ and

$$\left.\begin{aligned} a &= (1-A^2)^{-1/2} = \cosh\varphi \\ b &= A(1-A^2)^{-1/2} = \sinh\varphi \end{aligned}\right. \qquad \varphi > 0. \tag{3.19}$$

3.6.2. *MLT, repulsive case.* Here

$$\lambda > 0 \qquad \mu = 0 \qquad \theta = 0 \tag{3.20}$$

and, choosing $\theta_a = 0$, (3.14) requires that $\theta_b = -\pi$ and

$$a = (1 - A^2)^{-1/2} = \cosh\varphi \qquad \varphi < 0. \tag{3.21}$$
$$b = -A(1 - A^2)^{-1/2} = \sinh\varphi$$

In both cases,

$$\tanh 2\varphi = -\lambda/v_F \tag{3.22}$$

or

$$\tanh\varphi = -A \operatorname{sgn}\lambda \tag{3.22'}$$

which represent just the ML diagonalisation condition. So the MLT may be obtained from the general transformation as a particular case.

3.6.3. The $\mathcal{T}$ transformation. Here, the phase and the strength of the potential (θ and x) are no longer independent. It is useful to consider θ as a free variable, and x given by (2.7) and (3.15):

$$x = (\Lambda/v_F)\,[(2\cos\theta)/(1 + \cos^2\theta)] \tag{3.23}$$

In this case, from (2.3) and (3.15),

$$\gamma^2 = \cot^2\theta$$
$$A = \cos\theta > 0 \tag{3.24}$$

and, from (1.14),

$$a = 1 - i\gamma = (1/\sin\theta)\exp(-i\pi/2)\exp(i\theta)$$
$$b = \exp(-i\pi/2)\cot\theta. \tag{3.25}$$

So, $\theta_a = \theta - \pi/2$, $\theta_b = -\pi/2$ and the general diagonalisation conditions (3.13) and (3.14) are also fulfilled in this particular case.

4. Some useful particular transformations of H_{LM} and H_{GLM}

4.1. The effect of a MLT on H_{GLM}

It is easy to find that

$$\exp[iS_{ML}(\varphi)]H_{GLM}(v_F, \lambda, \mu)\exp[-iS_{ML}(\varphi)] = H_{GLM}(\tilde{v}_F, \tilde{\lambda}, \tilde{\mu}) + W \tag{4.1}$$

where H_{GDM} is given by (1.15) and

$$\tilde{v}_F = v_F\cosh 2\varphi + \lambda\sinh 2\varphi$$
$$\tilde{\lambda} = v_F\sinh 2\varphi + \lambda\cosh 2\varphi \tag{4.2}$$
$$\tilde{\mu} = \mu.$$

Here, φ is free of any constraint. W is the vacuum renormalisation energy, which is unimportant in this context.

If we choose for φ the value obtained by Luther and Emery (1974),

$$\exp\varphi = 1/\sqrt{2} \qquad \tanh 2\varphi = -\tfrac{3}{5} \tag{4.3}$$

we obtain

$$\begin{aligned}\tilde{v}_{\mathrm{F}} &= \tfrac{1}{4}(5v_{\mathrm{F}} - 3\lambda)\\ \tilde{\lambda} &= \tfrac{1}{4}(-3v_{\mathrm{F}} + 5\lambda)\\ \tilde{\mu} &= \mu.\end{aligned} \tag{4.4}$$

4.2. The effect of the $\mathcal{T}$ transformation on H_{LM}

Similarly

$$\exp[\mathrm{i}S_{\mathcal{T}}(\gamma)]H_{\mathrm{LM}}(v_{\mathrm{F}},\lambda)\exp[-\mathrm{i}S_{\mathcal{T}}(\gamma)] = H_{\mathrm{GLM}}(\tilde{v}_{\mathrm{F}},\tilde{\lambda},\tilde{\mu}) \tag{4.5}$$

where

$$\begin{aligned}\tilde{v}_{\mathrm{F}} &= v_{\mathrm{F}} + 2\gamma^2(v_{\mathrm{F}} - \lambda)\\ \tilde{\lambda} &= 2\gamma^2(v_{\mathrm{F}} - \lambda) + \lambda \qquad \gamma \in \mathbb{R}.\\ \tilde{\mu} &= -2\gamma(v_{\mathrm{F}} - \lambda)\end{aligned} \tag{4.6}$$

This Hamiltonian is not diagonal; if we try to impose on it the diagonalisation condition (2.3), i.e.

$$\tilde{\mu}^2 = 2\tilde{\lambda}(\tilde{v}_{\mathrm{F}} - \tilde{\lambda}) \tag{4.7}$$

we obtain $\lambda = v_{\mathrm{F}}$ (where diagonalisation is not possible) or $\lambda = 0$ (the free case). Of course, the Luttinger model cannot be diagonalised using a $\mathcal{T}$ transformation.

5. The physical behaviour of the GLM

In order to understand the physical behaviour of the GLM (1.15), let us compute its correlation functions at $T = 0$. We shall use the bosonisation method, in the version of Luther and Peschel (1974) and Mattis (1974). During the computation, a diagonalisation transformation is essentially used; specific Luttinger models require specific transformations (i.e. specific values of the coefficients a and b), as will be indicated below in equations (5.5) and (5.6). A slight momentum dependence of the couplings (as $\exp(-\alpha p)$) was assumed.

5.1. The Green functions

The Green functions are

$$\begin{aligned}G_1(x,t) &= -\mathrm{i} < T\,\psi_1(x,t)\psi_1^{\dagger}(0,0)\rangle\\ &= \operatorname{sgn} t\,[\exp(\mathrm{i}k_{\mathrm{F}}x)/2\pi][\xi + \mathrm{i}\alpha(t)]^{-1}\{[\xi + \mathrm{i}\alpha(t)][\zeta - \mathrm{i}\alpha(t)]\}^{-|b|^2}\end{aligned} \tag{5.1}$$

$$G_2(x,t) = -\operatorname{sgn} t\,[\exp(-\mathrm{i}k_{\mathrm{F}}x)/2\pi]\,[\zeta-\alpha(t)]^{-1}\{[\zeta-\mathrm{i}\alpha(t)][\xi+\mathrm{i}\alpha(t)]\}^{-|b|^2} \tag{5.2}$$

with the usual notation

$$\xi = x - \tilde{v}_{\mathrm{F}}t \qquad \zeta = x + \tilde{v}_{\mathrm{F}}t. \tag{5.3}$$

The exponent is

$$|b|^2 = A^2/(1-A^2) \tag{5.4}$$

independent of the phase of the potential.

For the standard Luttinger model, diagonalisable by a MLT,

$$|b|^2 = \sinh^2\varphi = [1-\sqrt{1-(\lambda/v_{\mathrm{F}})^2}]/2\sqrt{1-(\lambda/v_{\mathrm{F}})^2} \tag{5.5}$$

the known result.

For the GLM, defined by (1.15) and (1.13), diagonalisable by the $\mathcal{T}$ transformation, with equation (3.24), we find that

$$|b|^2 = \gamma^2 = \cot^2\theta \tag{5.6}$$

and the physical information is essentially the same as for the standard Luttinger model.

5.2. *The CDW and the SC correlation functions*

For the charge-density wave (CDW) correlation function

$$C(x,t) = \langle T\psi_1(x,t)\psi_2(x,t)\psi_2^{+}(0,0)\psi_1(0,0)\rangle \tag{5.7}$$

and for the pairing response, or superconducting (SC) function

$$S(x,t) = \langle T\psi_1(x,t)\psi_2(x,t)\psi_2^{+}(0,0)\psi_1^{+}(0,0)\rangle \tag{5.8}$$

we obtain

$$C(x,t) = [\exp(-2\mathrm{i}k_{\mathrm{F}}x)/(2\pi\alpha)^2]\{\alpha^2/[\xi+\mathrm{i}\alpha(t)][\zeta-\mathrm{i}\alpha(t)]\}^{c} \tag{5.9}$$

$$S(x,t) = (2\pi\alpha)^{-2}\{\alpha^2/[\xi+\mathrm{i}\alpha(t)][\zeta-\mathrm{i}\alpha(t)]\}^{s} \tag{5.10}$$

where the exponents c and s are functions of the potential (of Λ and θ) through a and b, according to (3.13)–(3.16):

$$c = |a|^2+|b|^2+ab+a^*b^* = (1-2A\cos\theta+A^2)/(1-A^2) \tag{5.11}$$

$$s = |a|^2+|b|^2-ab-a^*b^* = (1+2A\cos\theta+A^2)/(1-A^2). \tag{5.12}$$

Their Fourier transforms behave for $q\sim\omega\sim 0$ as

$$C(\omega)\sim\omega^{2c-2} \qquad S(\omega)\sim\omega^{2s-2}. \tag{5.13}$$

The CDW function becomes singular if

$$A - \cos\theta < 0 \tag{5.14}$$

or, equivalently,

$$(\lambda - v_{\mathrm{F}}/2)^2/(v_{\mathrm{F}}/2)^2 + \mu^2/(v_{\mathrm{F}}/\sqrt{2})^2 < 1 \tag{5.15}$$

i.e. for the region bounded by the ellipse defined in section 2.

The SC function $S(\omega)$ becomes singular if

$$A + \cos\theta < 0 \tag{5.16}$$

or, equivalently, if

$$(\lambda + v_F/2)^2/(v_F/2)^2 + \mu^2/(v_F/\sqrt{2})^2 < 1 \tag{5.17}$$

which is the region bounded by an ellipse with its centre at $(-v_F/2, 0)$, symmetric with (2.8) with respect to the μ axis (figure 2).

In particular, for the Luttinger model ($\mu = 0$), we obtain that the CDW (SC) function is singular in the repulsive (attractive) case, as is well known (Fogedby 1976).

So the GLM has the same states as the standard Luttinger model. The phase diagram of the GLM is drawn in figure 2.

6. The spin-dependent case

6.1. The model

Let us consider the case in which the fermionic field operators are spin dependent:

$$\psi_{js}(x) = \frac{1}{\sqrt{L}} \sum_k \exp(\mathrm{i}ks)\, a_{jks}. \tag{6.1}$$

The density operators are defined by

$$\rho_{js}(p) = \sum_k a^+_{j,k+p,s} a_{jks}. \tag{6.2}$$

We shall again study a generalised spin-dependent Luttinger model, described by the Hamiltonian

$$H = H_0 + H_{2\|} + H_{2\perp} + H_{\mathrm{BS}} + H_{\mathrm{US}}. \tag{6.3}$$

H_0 represents the kinetic part:

$$H_0 = v_F \left\{ \sum_{s;p>0} pa^+_{1ps}a_{1ps} + \sum_{s;p<0} p(a^+_{1ps}a_{1ps} - 1) - \sum_{s;p<0} pa^+_{2ps}a_{2ps} - \sum_{s;p>0} p(a^+_{2ps}a_{2ps} - 1)\right\}. \tag{6.4}$$

The next two terms describe the small-momentum-transfer scattering processes (forward-scattering terms); let us define them by the expressions

$$H_{2\|} = \frac{2\pi}{L} \sum_{s;k>0} [(g_{2\|} + \mathrm{i}h_{2\|})\rho^+_{2s}(k)\rho^+_{1s}(-k) + \mathrm{HC}) \qquad g_{2\|}, h_{2\|} \in \mathbb{R} \tag{6.5}$$

$$H_{2\perp} = \frac{2\pi}{L} \sum_{s;k>0} [(g_{2\perp} + \mathrm{i}h_{2\perp})\rho^+_{1s}(-k)\rho^+_{2,-s}(k) + \mathrm{HC}] \qquad g_{2\perp}, h_{2\perp} \in \mathbb{R}. \tag{6.6}$$

The last two terms in (6.3) represent the large-momentum-transfer processes, the BS and US interactions; they have the standard form

$$H_{\mathrm{BS}} = u \int \mathrm{d}x[\psi^+_{11}(x)\psi^+_{2-1}(x)\psi_{1-1}(x)\psi_{21}(x) + \mathrm{HC}] \tag{6.7}$$

$$H_{\mathrm{US}} = w \int \mathrm{d}x[\exp(\mathrm{i}Gx)\psi^+_{21}(x)\psi^+_{2-1}(x)\psi_{1-1}(x)\psi_{11}(x) + \mathrm{HC}]. \tag{6.8}$$

Some forward-scattering terms ($H_{3\|}$, $H_{3\perp}$ in Sòlyom's (1979) notation) were omitted because they do not contribute significantly to the physics of the system.

So the generalised spin-dependent Luttinger model defined by (6.23) differs from the standard one in the fact that the forward-scattering couplings are complex; the whole Hamiltonian remains Hermitian. This model can be exactly solved, if some restrictive conditions are imposed on the couplings. The rest of this section will be devoted to finding this solution.

6.2. *The spinless operator formalism*

Let us introduce as usual the spinless boson operators:

$$\rho_j = (1/\sqrt{2})(\rho_{j1} + \rho_{j-1}) \tag{6.9}$$

$$\sigma_j = (1/\sqrt{2})(\rho_{j1} - \rho_{j-1}). \tag{6.10}$$

So the sum of the first three terms in (6.3) may be separated as usual into a 'charge-dependent' and a 'spin-dependent' part:

$$H_0 + H_{2\|} + H_{2\perp} = H'_\rho + H'_\sigma \tag{6.11}$$

$$H'_\rho = \frac{2\pi v_{\mathrm{F}}}{L}\sum(\rho_1^+\rho_1 + \rho_2^+\rho_2) + \frac{2\pi}{L}\sum[(\lambda_\rho + \mathrm{i}\mu_\rho]\rho_1^+\rho_2^+ + \mathrm{HC}] \tag{6.12}$$

$$H'_\sigma = \frac{2\pi v_{\mathrm{F}}}{L}\sum(\sigma_1^+\sigma_1 + \sigma_2^+\sigma_2) + \frac{2\pi}{L}\sum[(\lambda_\sigma + \mathrm{i}\mu_\sigma)\sigma_1^+\sigma_2^+ + \mathrm{HC}] \tag{6.13}$$

where

$$\lambda_\rho = g_{2\|} + q_{2\perp} \qquad \mu_\rho = h_{2\|} + h_{2\perp} \tag{6.14}$$

$$\lambda_\sigma = g_{2\|} - g_{2\perp} \qquad \mu_\sigma = h_{2\|} - h_{2\perp}. \tag{6.15}$$

6.3. *The bosonisation scheme*

The large-momentum-transfer terms, H_{BS} and H_{US} ((6.7) and (6.8), respectively) may be treated in the Luther–Emery (1974) way. For instance, we find that

$$\begin{aligned}\psi_{11}^+(x)\psi_{2-1}^+(x)\psi_{1-1}(x)\psi_{21}(x) \\ &= (1/L^2)\exp[\sqrt{2}\Omega_{1\sigma}^+(x)]\exp[-\sqrt{2}\Omega_{1\sigma}(x)]\exp[-\sqrt{2}\Omega_{2\sigma}^+(x) \\ &\quad\times\exp[\sqrt{2}\Omega_{2\sigma}(x)]\end{aligned} \tag{6.16}$$

with

$$\Omega_{j\sigma}(x) = \frac{2\pi}{L}\sum_{k>0}\frac{\exp[-(\alpha/2)k + (-1)^j kx]}{k}\sigma_j((-1)^j k). \tag{6.17}$$

6.4. *The Luther–Emery transformation effect*

If we apply to the Hamiltonian

$$\begin{aligned}H_\sigma = {} & \frac{2\pi v_{\mathrm{F}}}{L}\sum(\sigma_1^+\sigma_1 + \sigma_2^+\sigma_2) + \frac{2\pi}{L}\sum[(\lambda_\sigma + \mathrm{i}\mu_\sigma)\sigma_1^+\sigma_2^+ + \mathrm{HC}] \\ & + \frac{u}{L^2}\int \mathrm{d}x\{\exp[\sqrt{2}\Omega_{1\sigma}^+(x)]\exp[-\sqrt{2}\Omega_{1\sigma}(x)]\exp[-\sqrt{2}\Omega_{2\sigma}^+(x)]\end{aligned}$$

$$\times \exp[\sqrt{2}\Omega_{2\sigma}(x)] + \mathrm{HC}\} \tag{6.18}$$

which contains all the spin degrees of freedom, a MLT transformation on σ operators with the Luther–Emery value

$$\exp \varphi = 1/\sqrt{2} \tag{6.19}$$

we find that

$$H_\sigma \xrightarrow[(6.19)]{\mathrm{MLT}} \tilde{H}_\sigma = \frac{2\pi v'_{F\sigma}}{L} \sum (\sigma_1^+ \sigma_1 + \sigma_2^+ \sigma_2) + \frac{2\pi}{L} \sum [(\lambda'_\sigma + \mathrm{i}\mu'_\sigma)\sigma_1^+ \sigma_2^+ + \mathrm{HC}]$$

$$+ \frac{u}{2\pi\alpha} \int \mathrm{d}x\, [\exp(2\,\mathrm{i}k_F x)\eta_1^+(x)\eta_2(x) + \mathrm{HC}]. \tag{6.20}$$

According to (4.4), the parameters entering (6.20) are

$$\begin{aligned} v'_{F\sigma} &= \tfrac{1}{4}(5v_F - 3\lambda_\sigma) \\ \lambda'_\sigma &= \tfrac{1}{4}(-3v_F + 5\lambda_\sigma) \\ \mu'_\sigma &= \mu_\sigma. \end{aligned} \tag{6.21}$$

η_1 and η_2 are the fermionic field operators associated with the bosonic spin-density operators

$$\begin{aligned} \eta_1(x) &= (1/\sqrt{L}) \exp(\mathrm{i}k_F x) \exp[-\Omega_{1\sigma}^+(x)] \exp[\Omega_{1\sigma}(x)] \\ \eta_2(x) &= (1/\sqrt{L}) \exp(-\mathrm{i}k_F x) \exp[-\Omega_{2\sigma}^+(x)] \exp[\Omega_{2\sigma}(x)]. \end{aligned} \tag{6.22}$$

6.5. *The $\mathcal{T}$ transformation effect*

Until now, the couplings in (6.20) which will diagonalise the bosonic term are free of any constraint. A $\mathcal{T}$ transformation applied to (6.20) will diagonalise the bosonic term and will leave the fermionic term unchanged if

$$\mu'^2_\sigma = 2\lambda'_\sigma(v_{F\sigma} - \lambda'_\sigma). \tag{6.23}$$

So,

$$\tilde{H}_\sigma \xrightarrow{\mathcal{T}} \tilde{\tilde{H}}_\sigma = \frac{2\pi v_{F\sigma}}{L} \sum (\sigma_1^+ \sigma_1 + \sigma_2^+ \sigma_2)$$

$$+ \frac{u}{2\pi\alpha} \int \mathrm{d}x\, [\exp(2\mathrm{i}k_F x)\, \eta_1^+(x)\eta_2(x) + \mathrm{HC}] \tag{6.24}$$

or

$$\tilde{\tilde{H}}_\sigma = v_{F\sigma} \sum_p p(c_{1p}^+ c_{1p} - c_{2p}^+ c_{2p}) + \frac{u}{2\pi\alpha} \sum_k (c_{1k} c_{2,k-2k_F}^+ + \mathrm{HC}) \tag{6.25}$$

where

$$v_{F\sigma} = v'_{F\sigma} - \lambda'_\sigma = 2(v_F - \lambda_\sigma). \tag{6.26}$$

The c operators are the Fourier components of the η operators:

$$\eta_j(x) = \frac{1}{\sqrt{L}} \sum_p \exp(ipx)\, c_{jp} \qquad j = 1, 2. \tag{6.27}$$

The Hamiltonian (6.25) is identical with that obtained by Luther and Emery (1974) in their solution of the BS problem. The correlation functions and the phase diagram of this Hamiltonian are known (Luther and Emery 1974, Lee 1975).

As a consequence of the restriction (6.23), the real part of the couplings must satisfy the constraint

$$\tfrac{3}{5} v_{\mathrm{F}} < g_{2\|} - g_{2\perp} < v_{\mathrm{F}}. \tag{6.28}$$

A similar treatment can be applied to the 'charge' degrees of freedom part of the Hamiltonian.

6.6. *Final remarks*

So the generalised Luttinger model can be completely solved, if some restrictions are imposed on the coupling constants. These are the following: the real part must satisfy the inequalities

$$\tfrac{3}{5} v_{\mathrm{F}} < g_{2\|} < v_{\mathrm{F}} \tag{6.29}$$

$$-\tfrac{1}{5} v_{\mathrm{F}} < g_{2\perp} < \tfrac{3}{5} v_{\mathrm{F}} \tag{6.30}$$

and the imaginary parts are fixed by the conditions

$$|h_{2\|} \pm h_{2\perp}| = (1/2\sqrt{2})\sqrt{34 v_{\mathrm{F}} \lambda_{\underset{\sigma}{\rho}} - 15(v_{\mathrm{F}} + \lambda_{\underset{\sigma}{\rho}})^2}. \tag{6.31}$$

Specifically, the generalised spin-dependent Luttinger Hamiltonian (6.3) was transformed into the sum of two bilinear Hamiltonians: one of these (equation (6.25)) collects the 'spin degrees of freedom'; the other is its 'charge' analogue. The physical behaviour of the system described by these Hamiltonians is known (Emery 1979).

7. Conclusions

In this paper, two bifermionic models whose interaction parts contain large-momentum-transfer terms are proposed. It was proved that this model can be exactly solved; indeed, it can be reduced to a bilinear Hamiltonian, as in the Luther–Emery solution, if some restrictions are imposed on the couplings. These restrictions are less severe than that required to obtain the exact solutions of the BS and US problems (Luther and Emery 1974, Emery *et al* 1975). It is specific to these models that the couplings entering the small-momentum-transfer terms are complex.

Do these results have more than a technical interest? One may wonder about the usefulness of a model with complex couplings. However, the Luttinger model is equivalent to a large number of other models; so the couplings are not necessarily the Fourier transforms of a direct-space (contact) potential. On the other hand, a number of one-dimensional Hamiltonians with complex 'couplings' have been examined in the literature (Black and Emery 1981, Voit and Schulz 1988, Martins 1988). We also hope that the physical relevance of our results can be better evaluated when renormalisation group calculations show which real models scale towards the Hamiltonians (2.9) and (6.3).

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